# ON Lie-CENTRAL EXTENSIONS OF LEIBNIZ ALGEBRAS

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**Abstract:** Basing ourselves on the categorical notions of central extensions and commutators in the framework of semi-abelian categories relative to a Birkhoff subcategory, we study central extensions of Leibniz algebras with respect to the Birkhoff subcategory of Lie algebras, called Lie-central extensions. We obtain a six-term exact homology sequence associated to a Lie-central extension. This sequence, together with the relative commutators, allows us to characterize several classes of Lie-central extensions, such as Lie-trivial, Lie-stem and Lie-stem cover, to introduce and characterize Lie-unicentral, Lie-capable, Lie-solvable and Lie-nilpotent Leibniz algebras.

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# 1 Introduction

A general theory of central extensions relative to a chosen subcategory of a base category was introduced in [15]. Recently, in [7] were analyzed in details the categorical concepts of central extensions, perfect objects and commutators in a semi-abelian category [2], relative to a Birkhoff subcategory. Some examples (e.g. groups vs. abelian groups, Lie (Leibniz) algebras vs. vector spaces) are absolute, meaning that they are fitted into the relative case with respect to the subcategory of all abelian objects. In the absolute case, some results were already investigated in [14].

In this paper we deal with the 'non absolute' case: Leibniz algebras vs. Lie algebras. In particular, the goal of the present paper is to consider the relative notions of central extension and commutator when the semi-abelian category is Leib, the category of Leibniz algebras, and its Birkhoff subcategory is Lie, the

category of Lie algebras, together with the Liezation functor $(-)_{\mathsf{Lie}} : \mathsf{Leib} \to \mathsf{Lie}$, which is left adjoint to the inclusion functor $\mathsf{Lie} \hookrightarrow \mathsf{Leib}$.

Under these circumstances, the notion of central extension relative to Lie, so called Lie-central extension, and the notion of commutator relative to Lie, provide the necessary ingredients to introduce the notions of unicentrality, capability, solvability and nilpotency of Leibniz algebras relative to Lie, all of them named with the prefix Lie-.

Homological machinery relative to Lie, coming from the categorical semi-abelian framework [11, 12, 13], allows us to characterize these new notions by means of the six-term exact sequence

$$\mathfrak{n} \otimes \mathfrak{g}_{\mathsf{Lie}} \longrightarrow HL_2^{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow HL_2^{\mathsf{Lie}}(\mathfrak{q}) \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{g}_{\mathsf{Lie}} \longrightarrow \mathfrak{q}_{\mathsf{Lie}} \longrightarrow 0$$

associated to the Lie-central extension $0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{q} \to 0$, where $HL_2^{\mathsf{Lie}}(-)$ denotes (the Hopf formula for) the second homology of a Leibniz algebra with coefficients in the Liezation functor $(-)_{\mathsf{Lie}}$ (see Subsections 3.1 and 3.2).

In particular, we organize the paper as follows: in Section 2 we recall the necessary categorical background from [7, 9]. Then we particularize in Subsection 3.1 these notions for the category Leib and its Birkhoff subcategory Lie. So we recall the concepts of Lie-perfect Leibniz algebras, Lie-trivial and Lie-central extensions of Leibniz algebras. In Subsection 3.2 we give the five-term exact sequence in homology with coefficients in the Liezation functor, associated to an extension of Leibniz algebras, and we obtain the above mentioned six-term exact sequence associated to a Lie-central extension. In Subsection 3.3 we introduce and study the notions of Lie-stem extension and Lie-stem cover of Leibniz algebras. In Section 4 we introduce the notions of Lie-unicentral, Lie-capable and precise Lie-center of Leibniz algebras and investigate interrelationships between them. In Section 5, by using the relative commutators, we introduce Lie-central and Lie-abelian series of Leibniz algebras. Then we introduce and study the concepts of Lie-solvability and Lie-nilpotency of Leibniz algebras. In Section 6 we obtain a homological characterization of Lie-nilpotency of a Leibniz algebra, which is similar to the Stallings' theorem for Leibniz algebras in the absolute case [6].

# 2 Categorical background

In this section we give an overview of needed definitions and results from [7, 8, 9] on central extensions, perfect objects and commutators in semi-abelian categories.

In their article [15], Janelidze and Kelly introduced a general theory of central extensions relative to a chosen subcategory $\mathcal{B}$ of the base category $\mathcal{A}$. This theory holds when $\mathcal{B}$ is a Birkhoff subcategory of a semi-abelian category $\mathcal{A}$ [9].

Recall from [16] that a category $\mathcal{A}$ is *semi-abelian* if it is pointed, Barr exact and Bourn protomodular with binary coproducts. A subcategory $\mathcal{B}$ of a semi-

abelian category $\mathcal{A}$ is *a Birkhoff subcategory* if it is full, reflective and closed under subobjects and regular quotients.

From now on, we consider $\mathcal{B}$ a fixed Birkhoff subcategory of a semi-abelian category $\mathcal{A}$. We denote by $I : \mathcal{A} \to \mathcal{B}$ the left adjoint to the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$, and write the components of its unit by $\eta_A : A \to I(A)$.

Then *an extension* in $\mathcal{A}$ is a regular epimorphism in $\mathcal{A}$.

An extension $f\colon B \twoheadrightarrow A$ in $\mathcal{A}$ is said to be *trivial (with respect to $\mathcal{B}$)* or *$\mathcal{B}$-trivial* [15] when the induced square

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "\eta_B"'] & A \arrow[d, "\eta_A"] \\ I(B) \arrow[r, "I(f)"'] & I(A) \end{tikzcd} \qquad (\mathbf{A})$$

is a pullback.

An extension $f\colon B \twoheadrightarrow A$ in $\mathcal{A}$ is called *central (with respect to $\mathcal{B}$)* or *$\mathcal{B}$-central* [15] when either one of the projections $f_0$, $f_1$ in the kernel pair $(R[f], f_0, f_1)$ of $f$ is $\mathcal{B}$-trivial. That is to say, $f$ is $\mathcal{B}$-central if and only if in the diagram

$$\begin{tikzcd} R[f] \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d, "\eta_{R[f]}"'] & B \arrow[r, "f"] \arrow[d, "\eta_B"] & A \\ IR[f] \arrow[r, shift left, "If_0"] \arrow[r, shift right, "If_1"'] & IB & \end{tikzcd}$$

either one of the left hand side squares is a pullback.

An object $P$ of $\mathcal{A}$ is called *perfect with respect to $\mathcal{B}$* or *$\mathcal{B}$-perfect* when $I(P)$ is the zero object 0 of $\mathcal{B}$.

Since, in a semi-abelian category, a regular epimorphism is always the cokernel of its kernel, an appropriate notion of short exact sequence exists. Such will be any sequence $K \xrightarrow{k} B \xrightarrow{f} A$ that satisfies $k = \mathrm{Ker}(f)$ and $f = \mathrm{Coker}(k)$. We denote this situation by $0 \longrightarrow K \xrightarrow{k} B \xrightarrow{f} A \longrightarrow 0$.

For any object $A$ of $\mathcal{A}$, the adjunction $\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "I"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "\supset"] \end{tikzcd}$ induces a short exact sequence

$$0 \longrightarrow [A, A]_{\mathcal{B}} \xrightarrow{\mu_A} A \xrightarrow{\eta_A} IA \longrightarrow 0.$$

Here the object $[A, A]_{\mathcal{B}}$, defined as the kernel of $\eta_A$, acts as *a zero-dimensional commutator relative to $\mathcal{B}$*. Of course, $IA = A/[A, A]_{\mathcal{B}}$, so that $A$ is an object of $\mathcal{B}$ if and only if $[A, A]_{\mathcal{B}}$ is zero. On the other hand, an object $A$ of $\mathcal{A}$ is called *$\mathcal{B}$-perfect* when $[A, A]_{\mathcal{B}} = A$.

Let us remark that an extension $f\colon B \twoheadrightarrow A$ in $\mathcal{A}$ is $\mathcal{B}$-central if and only if the restrictions $[f_0, f_0]_{\mathcal{B}}$, $[f_1, f_1]_{\mathcal{B}}\colon [R[f], R[f]]_{\mathcal{B}} \to [B, B]_{\mathcal{B}}$ of the kernel pair projections $f_0$, $f_1\colon R[f] \to B$ coincide. This is the case precisely when $[f_0, f_0]_{\mathcal{B}}$ and

$[f_1, f_1]_{\mathcal{B}}$ are isomorphisms, or, equivalently, when $\mathrm{Ker}[f_0, f_0]_{\mathcal{B}} \colon L_1[f] \to [R[f], R[f]]_{\mathcal{B}}$ is zero. In the following diagram

$$\begin{tikzcd}
 & L_1[f] \arrow[d, "\mathrm{Ker}[f_0,f_0]_{\mathcal{B}}"'] \arrow[drr, dotted] & & & & \\
0 \arrow[r] & {[R[f], R[f]]_{\mathcal{B}}} \arrow[r, "\mu_{R[f]}"] \arrow[d, shift right, "{[f_0,f_0]_{\mathcal{B}}}"'] \arrow[d, shift left, "{[f_1,f_1]_{\mathcal{B}}}"] & R[f] \arrow[r, "\eta_{R[f]}"] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "f_1"] & IR[f] \arrow[r] \arrow[d, shift right, "If_0"'] \arrow[d, shift left, "If_1"] & 0 \\
0 \arrow[r] & {[B, B]_{\mathcal{B}}} \arrow[r, "\mu_B"'] & B \arrow[r, "\eta_B"'] & IB \arrow[r] & 0 \, ,
\end{tikzcd}$$

through the composite $f_1 \circ \mu_{R[f]} \circ \mathrm{Ker}[f_0]_{\mathcal{B}}$ the object $L_1[f]$ may be considered as a normal subobject of $B$. It acts as *a one-dimensional commutator relative to* $\mathcal{B}$ and, if $K$ denotes the kernel of $f$, it is usually written $[K, B]_{\mathcal{B}}$.

Let $A$ be an object of a semi-abelian category $\mathcal{A}$ with enough projectives and $f \colon B \twoheadrightarrow A$ be a projective presentation with kernel $K$. The induced objects

$$\frac{[B, B]_{\mathcal{B}}}{[K, B]_{\mathcal{B}}} \qquad \text{and} \qquad \frac{K \cap [B, B]_{\mathcal{B}}}{[K, B]_{\mathcal{B}}}$$

are independent of the chosen projective presentation of $A$ as explained for instance in [12]. The object $(K \cap [B, B]_{\mathcal{B}})/[K, B]_{\mathcal{B}}$ is called (the Hopf formula for) the second homology object of $A$ (with coefficients in $\mathcal{B}$) and it is denoted by $H_2(A, I)$. We also write $H_1(A, I)$ for $I(A)$.

When $\mathcal{A}$ is a semi-abelian monadic category, the objects $H_1(A, I)$ and $H_2(A, I)$ may be computed using comonadic homology as in [13] and they are fitted into the semi-abelian homology theory (see [11]). Moreover, [12, Theorem 5.9] states that, given a short exact sequence $0 \longrightarrow K \longrightarrow B \longrightarrow A \longrightarrow 0$, there is the following five-term exact sequence

$$H_2(B, I) \longrightarrow H_2(A, I) \xrightarrow{\theta^*(B)} \frac{K}{[K, B]_{\mathcal{B}}} \longrightarrow H_1(B, I) \longrightarrow H_1(A, I) \longrightarrow 0 \, . \qquad \textbf{(B)}$$

# 3 Central extensions of Leibniz algebras with respect to Lie algebras

In this section we consider the particular case where $\mathcal{A}$ is the semi-abelian category $\mathsf{Leib}$ of Leibniz algebras, and the Birkhoff subcategory $\mathcal{B}$ is $\mathsf{Lie}$, the category of Lie algebras.

## 3.1 Preliminary results on Leibniz algebras

We fix $\mathbb{K}$ as a ground field such that $\frac{1}{2} \in \mathbb{K}$. All vector spaces and tensor products are considered over $\mathbb{K}$.

A *Leibniz algebra* [18, 19, 20] is a vector space $\mathfrak{q}$ equipped with a bilinear map $[-,-] : \mathfrak{q} \otimes \mathfrak{q} \to \mathfrak{q}$, usually called the *Leibniz bracket* of $\mathfrak{q}$, satisfying the *Leibniz identity*:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad x,y,z \in \mathfrak{q}.$$

Leibniz algebras form a semi-abelian category [7, 16], denoted by $\mathsf{Leib}$, whose morphisms are linear maps that preserve the Leibniz bracket.

A subalgebra $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{q}$ is said to be *left (resp. right) ideal* of $\mathfrak{q}$ if $[h,q] \in \mathfrak{h}$ (resp. $[q,h] \in \mathfrak{h}$), for all $h \in \mathfrak{h}$, $q \in \mathfrak{q}$. If $\mathfrak{h}$ is both left and right ideal, then $\mathfrak{h}$ is called *two-sided ideal* of $\mathfrak{q}$. In this case $\mathfrak{q}/\mathfrak{h}$ naturally inherits a Leibniz algebra structure.

For a Leibniz algebra $\mathfrak{q}$, we denote by $\mathfrak{q}^{\mathrm{ann}}$ the subspace of $\mathfrak{q}$ spanned by all elements of the form $[x,x]$, $x \in \mathfrak{q}$. Further, we consider

$$Z^r(\mathfrak{q}) = \{a \in \mathfrak{q} \mid [x,a] = 0,\ x \in \mathfrak{q}\}, \quad Z(\mathfrak{q}) = \{a \in \mathfrak{q} \mid [x,a] = 0 = [a,x],\ x \in \mathfrak{q}\}$$

and call the *right center* and *center* of $\mathfrak{q}$, respectively. It is proved in [17, Lemma 1.1] that both $\mathfrak{q}^{\mathrm{ann}}$ and $Z^r(\mathfrak{q})$ are two-sided ideals of $\mathfrak{q}$. It is obvious that $Z(\mathfrak{q})$ is also a two-sided ideal of $\mathfrak{q}$.

Given a Leibniz algebra $\mathfrak{q}$, it is clear that the quotient $\mathfrak{q}_{\mathrm{Lie}} = \mathfrak{q}/\mathfrak{q}^{\mathrm{ann}}$ is a Lie algebra. This defines the so-called *Liezation functor* $(-)_{\mathsf{Lie}} : \mathsf{Leib} \to \mathsf{Lie}$, which assigns to a Leibniz algebra $\mathfrak{q}$ the Lie algebra $\mathfrak{q}_{\mathrm{Lie}}$. Moreover, the canonical epimorphism $\mathfrak{q} \twoheadrightarrow \mathfrak{q}_{\mathrm{Lie}}$ is universal among all homomorphisms from $\mathfrak{q}$ to a Lie algebra, implying that the Liezation functor is left adjoint to the inclusion functor $\mathsf{Lie} \hookrightarrow \mathsf{Leib}$.

It is an easy task to check that the category $\mathsf{Lie}$ is a Birkhoff subcategory of $\mathsf{Leib}$. Focusing our attention in the adjoint pair

$$\begin{tikzcd} \mathsf{Leib} \arrow[r, shift left, "(-)_{\mathsf{Lie}}"] \arrow[r, phantom, "\perp"] & \mathsf{Lie} \arrow[l, shift left, hook', "\supset"] \end{tikzcd}, \tag{C}$$

we particularize the general theory described in Section 2 to the case when $\mathcal{A}$ is the category $\mathsf{Leib}$, $\mathcal{B}$ is its Birkhoff subcategory $\mathsf{Lie}$ and the functor $I$ is precisely the Liezation functor $(-)_{\mathsf{Lie}}$. First let us remark the following

**Remark 3.1** *A Leibniz algebra $\mathfrak{q}$ is* $\mathsf{Lie}$*-perfect if $\mathfrak{q}_{\mathsf{Lie}} = 0$, that is $\mathfrak{q} \cong \mathfrak{q}^{\mathrm{ann}}$. It follows by [5, Lemma 3.1] that any* $\mathsf{Lie}$*-perfect Leibniz algebra $\mathfrak{q}$ is the trivial one. Then, by [9, Theorem 3.5], we get that a Leibniz algebra $\mathfrak{q}$ admits a universal* $\mathsf{Lie}$*-central extension if and only if $\mathfrak{q} = 0$.*

It is clear that an extension in $\mathsf{Leib}$ is just an epimorphism $f\colon \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ of Leibniz algebras, which is a $\mathsf{Lie}$-trivial extension if and only if it induces an isomorphism $\mathfrak{g}^{\mathrm{ann}} \cong \mathfrak{q}^{\mathrm{ann}}$.

**Example 3.2** *Let $\mathfrak{g}$ and $\mathfrak{q}$ be three and two-dimensional (as vector spaces) Leibniz algebras with $\mathbb{K}$-linear bases $\{a_1, a_2, a_3\}$ and $\{e_1, e_2\}$, respectively, with the Leibniz brackets given respectively by $[a_1, a_3] = a_1$ and $[e_1, e_2] = e_1$ and zero elsewhere (see [5, 10]). Consider the homomorphism of Leibniz algebras $f : \mathfrak{g} \to \mathfrak{q}$ defined by $f(a_1) = e_1$, $f(a_2) = 0$, $f(a_3) = e_2$. Obviously $f$ is surjective, $\mathfrak{g}^{\mathrm{ann}} = \langle \{a_1\} \rangle$ and $\mathfrak{q}^{\mathrm{ann}} = \langle \{e_1\} \rangle$, so they are isomorphic through $f$. Consequently $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ is a $\mathsf{Lie}$-trivial extension.*

For a Leibniz algebra $\mathfrak{q}$ and two-sided ideals $\mathfrak{m}$ and $\mathfrak{n}$ of $\mathfrak{q}$, we put

$$C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n}) = \{q \in \mathfrak{q} \mid \ [q, m] + [m, q] \in \mathfrak{n}, \text{ for all } m \in \mathfrak{m}\} \ .$$

Further, we denote by $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}}$ the subspace of $\mathfrak{q}$ spanned by all elements of the form $[m, n] + [n, m]$, $m \in \mathfrak{m}$, $n \in \mathfrak{n}$.

**Lemma 3.3** *Let $\mathfrak{q}$ be a Leibniz algebra and $\mathfrak{m}$, $\mathfrak{n}$ be two-sided ideals of $\mathfrak{q}$. Then both $C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n})$ and $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}}$ are two-sided ideals of $\mathfrak{q}$. Moreover, $Z(\mathfrak{q}) \subseteq C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n})$ and $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}} \subseteq Z^r(\mathfrak{q})$.*

*Proof.* It is clear that both $C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n})$ and $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}}$ are subspaces of $\mathfrak{q}$.

Take any elements $q \in C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n})$ and $x \in \mathfrak{q}$. Then, for any $m \in \mathfrak{m}$, we have

$$\begin{aligned} [[q, x], m] + [m, [q, x]] &= -[q, [m, x]] + [[q, m], x] + [[m, q], x] - [[m, x], q] \\ &= -\big([q, [m, x]] + [[m, x], q]\big) + \big[[q, m] + [m, q], x\big] \in \mathfrak{n} \end{aligned}$$

and, if we denote $-\big([q, [m, x]] + [[m, x], q]\big) + \big[[q, m] + [m, q], x\big]$ by $n$, we get

$$\begin{aligned} [[x, q], m] + [m, [x, q]] &= [[x, q], m] - [m, [q, x]] = [[x, q], m] - n + [[q, x], m] \\ &= \big[[x, q] + [q, x], m\big] - n \in \mathfrak{n}. \end{aligned}$$

Consequently, $C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n})$ is a two-sided ideal of $\mathfrak{q}$. The inclusion $Z(\mathfrak{q}) \subseteq C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{m}, \mathfrak{n})$ is obvious.

Now, for any $m \in \mathfrak{m}$, $n \in \mathfrak{n}$ and $q \in \mathfrak{q}$, we have

$$[[m, n] + [n, m], q] = [m, [n, q]] + [[n, q], m] + [n, [m, q]] + [[m, q], n] \in [\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}},$$

and

$$[q, [m, n] + [n, m]] = 0.$$

Consequently, $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}}$ is a two-sided ideal of $\mathfrak{q}$ contained in $Z^r(\mathfrak{q})$. $\square$

In particular, the two-sided ideal $C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{q}, 0)$ is the $\mathsf{Lie}$-*center* of the Leibniz algebra $\mathfrak{q}$ and it will be denoted by $Z_{\mathsf{Lie}}(\mathfrak{q})$, that is,

$$Z_{\mathsf{Lie}}(\mathfrak{q}) = \{z \in \mathfrak{q} \mid [q, z] + [z, q] = 0 \text{ for all } q \in \mathfrak{q}\}.$$

The following proposition is an immediate consequence of the discussion in [7, Example 1.9].

**Proposition 3.4** *Given an extension of Leibniz algebras $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ with $\mathfrak{n} = \mathrm{Ker}(f)$, the following conditions are equivalent:*

(*a*) $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ *is* Lie-*central;*

(*b*) $\mathfrak{n} \subseteq Z_{\mathsf{Lie}}(\mathfrak{q})$*;*

(*c*) $[\mathfrak{n}, \mathfrak{g}]_{\mathsf{Lie}} = 0$.

**Remark 3.5** *Obviously every central extension $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ (i.e. $\mathrm{Ker}(f) \subseteq Z(\mathfrak{g})$) is a* Lie-*central extension, but the converse is not true in general as the following example shows: consider the three-dimensional Leibniz algebra $\mathfrak{g}$ with $\mathbb{K}$-linear basis $\{a_1, a_2, a_3\}$, the Leibniz bracket given by $[a_1, a_3] = a_1$ and zero elsewhere, and the two-dimensional abelian Leibniz algebra $\mathfrak{q}$ with $\mathbb{K}$-linear basis $\{e_1, e_2\}$. Then the homomorphism of Leibniz algebras $f : \mathfrak{g} \to \mathfrak{q}$ given by $f(a_1) = 0, f(a_2) = e_1, f(a_3) = e_2$ is surjective, $\mathrm{Ker}(f) = \langle\{a_1\}\rangle$, $Z(\mathfrak{g}) = \langle\{a_2\}\rangle$ and $Z_{\mathsf{Lie}}(\mathfrak{g}) = \langle\{a_1, a_2\}\rangle$. Consequently the extension $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ is not central, but it is* Lie-*central.*

## 3.2 Six-term exact sequence

In what follows, given a Leibniz algebra $\mathfrak{g}$, we write $HL_2^{\mathsf{Lie}}(\mathfrak{g})$ for $H_2(\mathfrak{g}, (-)_{\mathsf{Lie}})$ and $HL_1^{\mathsf{Lie}}(\mathfrak{g}))$ for $H_1(\mathfrak{g}, (-)_{\mathsf{Lie}})$. Thus, the five-term exact sequence (**B**) associated to an extension of Leibniz algebras $f\colon \mathfrak{g} \twoheadrightarrow \mathfrak{q}$, with $\mathfrak{n} = \mathrm{Ker}(f)$, turns to

$$HL_2^{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow HL_2^{\mathsf{Lie}}(\mathfrak{q}) \xrightarrow{\theta^*(\mathfrak{g})} \frac{\mathfrak{n}}{[\mathfrak{n}, \mathfrak{g}]_{\mathsf{Lie}}} \longrightarrow HL_1^{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow HL_1^{\mathsf{Lie}}(\mathfrak{q}) \longrightarrow 0. \qquad (\mathbf{D})$$

Furthermore, we have the following proposition.

**Proposition 3.6** *Given a* Lie-*central extension $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ of Leibniz algebras with $\mathfrak{n} = \mathrm{Ker}(f)$, there is a six-term exact sequence*

$$\mathfrak{n} \otimes \mathfrak{g}_{\mathsf{Lie}} \longrightarrow HL_2^{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow HL_2^{\mathsf{Lie}}(\mathfrak{q}) \xrightarrow{\theta^*(\mathfrak{g})} \mathfrak{n} \longrightarrow HL_1^{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow HL_1^{\mathsf{Lie}}(\mathfrak{q}) \longrightarrow 0. \qquad (\mathbf{E})$$

*Proof.* Any free presentation of $\mathfrak{g}$, $0 \to \mathfrak{r} \to \mathfrak{f} \xrightarrow{\rho} \mathfrak{g} \to 0$, produces a free presentation of $\mathfrak{q}$, $0 \to \mathfrak{s} \to \mathfrak{f} \xrightarrow{\tau = f \circ \rho} \mathfrak{q} \to 0$, and the following commutative

diagram of Leibniz algebras

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[dl] & \\
& & \mathfrak{r} \arrow[dl] \arrow[d] & & \\
0 \arrow[r] & \mathfrak{s} \arrow[r] \arrow[d] & \mathfrak{f} \arrow[d, "\rho"] \arrow[dr, "\tau=f\circ\rho"] & & \\
0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r, "f"] \arrow[d] & \mathfrak{q} \arrow[r] \arrow[dr] & 0 \\
& 0 & 0 & & 0
\end{tikzcd} \tag{$\mathbf{F}$}$$

Thus we have

$$\operatorname{Ker}\left(HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q})\right) \approx \operatorname{Ker}\left(\frac{\mathfrak{r}\cap[\mathfrak{f},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}} \to \frac{\mathfrak{s}\cap[\mathfrak{f},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right) \approx \frac{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}.$$

When $f$ is a $\mathsf{Lie}$-central extension, $[\mathfrak{n},\mathfrak{g}]_{\mathsf{Lie}} = 0$, and by the exact sequence ($\mathbf{D}$) we get the following exact sequence

$$0 \longrightarrow \frac{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}} \longrightarrow \frac{\mathfrak{r}\cap[\mathfrak{f},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}} \longrightarrow \frac{\mathfrak{s}\cap[\mathfrak{f},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}} \xrightarrow{\theta^*(\mathfrak{g})} \mathfrak{n} \longrightarrow \mathfrak{g}_{\mathsf{Lie}} \longrightarrow \mathfrak{q}_{\mathsf{Lie}} \longrightarrow 0. \tag{$\mathbf{G}$}$$

Now consider $\sigma : \mathfrak{n}\otimes\mathfrak{g}_{\mathsf{Lie}} \to \frac{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}$ given by $\sigma(n\otimes\overline{x}) = [s,f]+[f,s]+[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}$, where $s\in\mathfrak{r}$ and $f\in\mathfrak{f}$ such that $\rho(s)=n$ and $\rho(f)=x$. Again using the condition $[\mathfrak{n},\mathfrak{g}]_{\mathsf{Lie}}=0$, it is easy to check that $\sigma$ is a well-defined epimorphism. Then the assertion follows. $\Box$

## 3.3 Some properties of $\mathsf{Lie}$-central extensions

In this subsection we establish some properties of $\mathsf{Lie}$-central extensions of Leibniz algebras by using the six-term exact sequence ($\mathbf{E}$) associated to the given extension.

**Proposition 3.7** *Let $f\colon \mathfrak{g}\twoheadrightarrow\mathfrak{q}$ be an extension of Leibniz algebras with $\mathfrak{n}=$* Ker$(f)$*. The following statements are equivalent:*

*(a)* $f\colon\mathfrak{g}\twoheadrightarrow\mathfrak{q}$ *is a* $\mathsf{Lie}$*-trivial extension.*

*(b)* $\mathfrak{n}\cap\mathfrak{g}^{\mathrm{ann}}=0$.

*(c)* $0\to\mathfrak{n}\to\mathfrak{g}_{\mathsf{Lie}}\to\mathfrak{q}_{\mathsf{Lie}}\to 0$ *is exact in* $\mathsf{Lie}$.

*Proof.* The equivalences follow from the definition of $\mathsf{Lie}$-trivial extension and by the $3 \times 3$ Lemma [3] applied to the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{n} \cap \mathfrak{g}^{\mathrm{ann}} \arrow[r] \arrow[d] & \mathfrak{g}^{\mathrm{ann}} \arrow[r] \arrow[d] & \mathfrak{q}^{\mathrm{ann}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\mathfrak{n}}{\mathfrak{n} \cap \mathfrak{g}^{\mathrm{ann}}} \arrow[r] \arrow[d] & HL_1^{\mathsf{Lie}}(\mathfrak{g}) \arrow[r] \arrow[d] & HL_1^{\mathsf{Lie}}(\mathfrak{q}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

**Proposition 3.8** *Let* $f \colon \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ *be a* $\mathsf{Lie}$*-trivial extension with* $\mathfrak{n} = \mathrm{Ker}(f)$*, then* $\theta^*(\mathfrak{g}) : HL_2^{\mathsf{Lie}}(\mathfrak{q}) \to \mathfrak{n}$ *is the zero map and* $\mathfrak{n} \otimes \mathfrak{g}_{\mathsf{Lie}} \to HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q}) \to 0$ *is exact.*

*Proof.* Since $[\mathfrak{n}, \mathfrak{g}]_{\mathsf{Lie}} \subseteq \mathfrak{n} \cap \mathfrak{g}^{\mathrm{ann}}$, by Proposition 3.7 we have that $[\mathfrak{n}, \mathfrak{g}]_{\mathsf{Lie}} = 0$, then $\mathrm{Im}(\theta^*(\mathfrak{g})) = 0$ in sequence (**E**). □

**Corollary 3.9** *Let* $f \colon \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ *be a* $\mathsf{Lie}$*-trivial extension with* $\mathfrak{n} = \mathrm{Ker}(f)$*, then* $\mathfrak{n} = Z_{\mathsf{Lie}}(\mathfrak{n})$*.*

*Proof.* Since $\mathfrak{n} \cap \mathfrak{g}^{\mathrm{ann}} = 0$, we have $[n, n'] + [n', n] = [n + n', n + n'] = 0$ for any $n, n' \in \mathfrak{n}$, which means that $\mathfrak{n} \subseteq Z_{\mathrm{Lie}}(\mathfrak{n})$. □

**Definition 3.10** *A* $\mathsf{Lie}$*-central extension* $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ *is said to be:*

*(a) a* $\mathsf{Lie}$*-stem extension if* $\mathfrak{g}_{\mathsf{Lie}} \cong \mathfrak{q}_{\mathsf{Lie}}$*.*

*(b) a* $\mathsf{Lie}$*-stem cover if* $\mathfrak{g}_{\mathsf{Lie}} \cong \mathfrak{q}_{\mathsf{Lie}}$ *and the induced map* $HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q})$ *is the zero map.*

**Example 3.11**

*(a) Let* $\mathfrak{g}$ *and* $\mathfrak{q}$ *be three and two-dimensional Leibniz algebra with* $\mathbb{K}$*-linear bases* $\{a_1, a_2, a_3\}$ *and* $\{e_1, e_2\}$*, with the Leibniz brackets given respectively by* $[a_2, a_3] = -[a_3, a_2] = a_2$*,* $[a_3, a_3] = a_1$ *and* $[e_1, e_2] = -[e_2, e_1] = -e_1$ *and zero elsewhere. Then the surjective homomorphism of Leibniz algebras* $f : \mathfrak{g} \to \mathfrak{q}$ *defined by* $f(a_1) = 0$*,* $f(a_2) = e_1$ *and* $f(a_3) = e_2$ *is a* $\mathsf{Lie}$*-stem extension.*

*(b) Let $0 \to \mathfrak{s} \to \mathfrak{f} \xrightarrow{\tau} \mathfrak{q} \to 0$ be a free presentation of a Leibniz algebra $\mathfrak{q}$. Then $[\mathfrak{s}, \mathfrak{f}]_{\mathsf{Lie}}$ is a two-sided ideal of $\mathfrak{f}$ and the $3 \times 3$ Lemma [3] provides the sequence $0 \to \mathfrak{s}/[\mathfrak{s}, \mathfrak{f}]_{\mathsf{Lie}} \to \mathfrak{f}/[\mathfrak{s}, \mathfrak{f}]_{\mathsf{Lie}} \to \mathfrak{q} \to 0$ which is a* Lie*-stem cover of $\mathfrak{q}$.*

**Proposition 3.12** *For a* Lie*-central extension $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$, with $\mathfrak{n} = \mathrm{Ker}(f)$, the following statements are equivalent:*

*(a) $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ is a* Lie*-stem extension.*

*(b) The induced map $\mathfrak{n} \to HL_1^{\mathsf{Lie}}(\mathfrak{q})$ is the zero map.*

*(c) $\theta^*(\mathfrak{g}) : HL_2^{\mathsf{Lie}}(\mathfrak{q}) \to \mathfrak{n}$ is an epimorphism.*

*(d) The following sequence $\mathfrak{n} \otimes \mathfrak{g}_{\mathsf{Lie}} \to HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q}) \xrightarrow{\theta^*(\mathfrak{g})} \mathfrak{n} \to 0$ is exact.*

*(e) $\mathfrak{n} \subseteq \mathfrak{g}^{\mathrm{ann}}$.*

*Proof.* The equivalences between (a), (b), (c) and (d) follow from the exact sequence (**E**). The equivalence between (a) and (e) is a consequence of the following $3 \times 3$ diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{n} \cap \mathfrak{g}^{\mathrm{ann}} \arrow[r] \arrow[d] & \mathfrak{g}^{\mathrm{ann}} \arrow[r] \arrow[d] & \mathfrak{q}^{\mathrm{ann}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & HL_1^{\mathsf{Lie}}(\mathfrak{g}) \arrow[r, equal] \arrow[d] & HL_1^{\mathsf{Lie}}(\mathfrak{q}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

**Proposition 3.13** *For a* Lie*-central extension $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ the following statements are equivalent:*

*(a) $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ is a* Lie*-stem cover.*

*(b) $\theta^*(\mathfrak{g}) : HL_2^{\mathsf{Lie}}(\mathfrak{q}) \to \mathfrak{n}$ is an isomorphism.*

*Proof.* This is a direct consequence of the six-term exact sequence (**E**). □

# 4 The precise $\mathsf{Lie}$-center of a Leibniz algebra

In this section we introduce the notions of $\mathsf{Lie}$-unicentrality, $\mathsf{Lie}$-capability and precise $\mathsf{Lie}$-center of a Leibniz algebra and analyze the relationships between them.

**Definition 4.1** *A Leibniz algebra $\mathfrak{q}$ is said to be $\mathsf{Lie}$-unicentral if every $\mathsf{Lie}$-central extension $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ satisfies $f(Z_{\mathsf{Lie}}(\mathfrak{g})) = Z_{\mathsf{Lie}}(\mathfrak{q})$, that is, the following diagram with exact rows*

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, hook] \arrow[d, equal] & Z_{\mathsf{Lie}}(\mathfrak{g}) \arrow[r, two heads] \arrow[d, hook] & Z_{\mathsf{Lie}}(\mathfrak{q}) \arrow[d, hook] \\ \mathrm{Ker}(f) \arrow[r, hook] & \mathfrak{g} \arrow[r, two heads, "f"] & \mathfrak{q} \end{tikzcd}$$

*is commutative.*

**Definition 4.2** *A Leibniz algebra $\mathfrak{q}$ is said to be $\mathsf{Lie}$-capable if there exists a $\mathsf{Lie}$-central extension*

$$0 \longrightarrow Z_{\mathsf{Lie}}(\mathfrak{g}) \longrightarrow \mathfrak{g} \xrightarrow{f} \mathfrak{q} \longrightarrow 0.$$

**Definition 4.3** *The precise $\mathsf{Lie}$-center $Z^*_{\mathsf{Lie}}(\mathfrak{q})$ of a Leibniz algebra $\mathfrak{q}$ is the intersection of all two-sided ideals $f(Z_{\mathsf{Lie}}(\mathfrak{g}))$, where $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ is a $\mathsf{Lie}$-central extension.*

**Remark 4.4**

*(a)* $Z^*_{\mathsf{Lie}}(\mathfrak{q}) \subseteq Z_{\mathsf{Lie}}(\mathfrak{q})$.

*(b)* $Z^*_{\mathsf{Lie}}(\mathfrak{q}) = Z_{\mathsf{Lie}}(\mathfrak{q})$ *if and only if* $f(Z_{\mathsf{Lie}}(\mathfrak{g})) = Z_{\mathsf{Lie}}(\mathfrak{q})$ *for every $\mathsf{Lie}$-central extension of Leibniz algebras $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$, or equivalently, if and only if $\mathfrak{q}$ is $\mathsf{Lie}$-unicentral.*

Given a free presentation $0 \to \mathfrak{s} \to \mathfrak{f} \xrightarrow{\tau} \mathfrak{q} \to 0$ of the Leibniz algebra $\mathfrak{q}$, consider the $\mathsf{Lie}$-central extension ($\mathsf{Lie}$-stem cover of $\mathfrak{q}$)

$$0 \longrightarrow \frac{\mathfrak{s}}{[\mathfrak{s}, \mathfrak{f}]_{\mathsf{Lie}}} \longrightarrow \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\mathsf{Lie}}} \xrightarrow{\overline{\tau}} \mathfrak{q} \longrightarrow 0$$

as in Example 3.11 (b). Then we have

**Lemma 4.5** $Z^*_{\mathsf{Lie}}(\mathfrak{q}) = \overline{\tau}\left(Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right)\right)$.

*Proof.* We need to show that, for any Lie-central extension $0 \to \mathfrak{a} \to \mathfrak{h} \xrightarrow{\varphi} \mathfrak{q} \to 0$, the inclusion $\overline{\tau}\left(Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right)\right) \subseteq \varphi\left(Z_{\mathsf{Lie}}(\mathfrak{h})\right)$ holds.

Since $\mathfrak{f}$ is a free Leibniz algebra, there exists (uniquely defined epimorphism) $\alpha : \mathfrak{f} \to \mathfrak{h}$ such that $\varphi \circ \alpha = \tau$. Then $\alpha(\mathfrak{s}) \subseteq \mathfrak{a}$ and $\alpha\left([\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}\right) \subseteq [\mathfrak{a},\mathfrak{h}]_{\mathrm{Lie}} = 0$. Hence, $\alpha$ induces $\overline{\alpha} : \frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}} \to \mathfrak{h}$ such that $\overline{\alpha} \circ \pi = \alpha$, where $\pi : \mathfrak{f} \twoheadrightarrow \frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}$ is the natural projection. It is straightforward to see that $\overline{\alpha}\left(Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right)\right) \subseteq Z_{\mathsf{Lie}}(\mathfrak{h})$. Now, since $\overline{\tau} \circ \pi = \tau = \varphi \circ \alpha = \varphi \circ \overline{\alpha} \circ \pi$, it follows that

$$\overline{\tau}\left(Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right)\right) = (\varphi \circ \overline{\alpha})\left(Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right)\right) \subseteq \varphi\left(Z_{\mathsf{Lie}}(\mathfrak{h})\right).$$

□

**Corollary 4.6** *$Z^*_{\mathsf{Lie}}(\mathfrak{q}) = 0$ if and only if $\mathfrak{q}$ is a Lie-capable Leibniz algebra.*

*Proof.* If $\mathfrak{q}$ is a Lie-capable Leibniz algebra, then there exists a Lie-central extension $0 \to Z_{\mathsf{Lie}}(\mathfrak{g}) \to \mathfrak{g} \xrightarrow{f} \mathfrak{q} \to 0$, then $Z^*_{\mathsf{Lie}}(\mathfrak{q}) \subseteq f\left(Z_{\mathsf{Lie}}(\mathfrak{g})\right) = 0$.

Conversely, if $Z^*_{\mathsf{Lie}}(\mathfrak{q}) = 0$, for any free presentation $0 \to \mathfrak{s} \to \mathfrak{f} \xrightarrow{\tau} \mathfrak{q} \to 0$ of $\mathfrak{q}$, we have $\overline{\tau}\left(Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right)\right) = 0$ by Lemma 4.5. Then

$$0 \longrightarrow Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}\right) \longrightarrow \frac{\mathfrak{f}}{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}} \xrightarrow{\overline{\tau}} \mathfrak{q} \longrightarrow 0$$

is a Lie-central extension. □

**Lemma 4.7** *Let $\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ be an epimorphism of Leibniz algebras, then $\pi\left(Z^*_{\mathsf{Lie}}(\mathfrak{g})\right) \subseteq Z^*_{\mathsf{Lie}}(\mathfrak{q})$.*

*Proof.* For any Lie-central extension $0 \to \mathfrak{n} \to \mathfrak{h} \xrightarrow{\varphi} \mathfrak{q} \to 0$ of $\mathfrak{q}$, consider the pull-back diagram over $\pi$

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d, equal] & \mathfrak{g} \times_{\mathfrak{q}} \mathfrak{h} \arrow[r, "\overline{\varphi}"] \arrow[d, two heads, "\overline{\pi}"] & \mathfrak{g} \arrow[r] \arrow[d, two heads, "\pi"] & 0 \\
0 \arrow[r] & \mathfrak{n} \arrow[r] & \mathfrak{h} \arrow[r, "\varphi"] & \mathfrak{q} \arrow[r] & 0 .
\end{tikzcd}$$

Clearly the upper row is again a Lie-central extension of $\mathfrak{q}$. Then we have $\pi\left(Z^*_{\mathsf{Lie}}(\mathfrak{g})\right) \subseteq \pi \circ \overline{\varphi}\left(Z_{\mathsf{Lie}}(\mathfrak{g} \times_{\mathfrak{q}} \mathfrak{h})\right) = \varphi \circ \overline{\pi}\left(Z_{\mathsf{Lie}}(\mathfrak{g} \times_{\mathfrak{q}} \mathfrak{h})\right) \subseteq \varphi\left(Z_{\mathsf{Lie}}(\mathfrak{h})\right)$. This implies that $\pi\left(Z^*_{\mathsf{Lie}}(\mathfrak{g})\right) \subseteq \bigcap\limits_{\varphi} \varphi\left(Z_{\mathsf{Lie}}(\mathfrak{h})\right) = Z^*_{\mathsf{Lie}}(\mathfrak{q})$. □

**Proposition 4.8**

(a) *$Z^*_{\mathsf{Lie}}(\mathfrak{q})$ is the smallest two-sided ideal $\mathfrak{n}$ of the Leibniz algebra $\mathfrak{q}$ such that $\mathfrak{q}/\mathfrak{n}$ is Lie-capable. In particular, $\mathfrak{q}/Z^*_{\mathsf{Lie}}(\mathfrak{q})$ is Lie-capable.*

(b) *Let $\mathfrak{n}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{q}$ such that $\mathfrak{n} \cap Z^*_{\mathsf{Lie}}(\mathfrak{q}) = 0$. If $\mathfrak{q}/\mathfrak{n}$ is* Lie*-capable, then $\mathfrak{q}$ is* Lie*-capable as well.*

*Proof.* (a) Given a two-sided ideal $\mathfrak{n}$ of $\mathfrak{q}$ such that $\mathfrak{q}/\mathfrak{n}$ is Lie-capable, consider the epimorphism $\pi : \mathfrak{q} \twoheadrightarrow \mathfrak{q}/\mathfrak{n}$. By Lemma 4.7 and Corollary 4.6 we have $\pi\left(Z^*_{\mathsf{Lie}}(\mathfrak{q})\right) \subseteq Z^*_{\mathsf{Lie}}(\mathfrak{q}/\mathfrak{n}) = 0$, then $Z^*_{\mathsf{Lie}}(\mathfrak{q}) \subseteq \mathrm{Ker}(\pi) = \mathfrak{n}$.

(b) By (a), $Z^*_{\mathsf{Lie}}(\mathfrak{q}) \subseteq \mathfrak{n}$. Since $Z^*_{\mathsf{Lie}}(\mathfrak{q}) \cap \mathfrak{n} = 0$, then $Z^*_{\mathsf{Lie}}(\mathfrak{q}) = 0$. Then Corollary 4.6 completes the proof. $\square$

**Theorem 4.9** *Let $\mathfrak{a}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{q}$ such that $\mathfrak{a} \subseteq Z_{\mathsf{Lie}}(\mathfrak{q})$. Then $\mathfrak{a} \subseteq Z^*_{\mathsf{Lie}}(\mathfrak{q})$ if and only if the map $C : \mathfrak{a} \otimes \mathfrak{q}_{\mathsf{Lie}} \to HL_2^{\mathsf{Lie}}(\mathfrak{q})$ in sequence* (**E**) *associated to the* Lie*-central extension $0 \to \mathfrak{a} \to \mathfrak{q} \stackrel{\pi}{\to} \mathfrak{q}/\mathfrak{a} \to 0$ is the zero map.*

*Proof.* Consider the free presentations $0 \to \mathfrak{r} \to \mathfrak{f} \stackrel{\rho}{\to} \mathfrak{q} \to 0$ and $0 \to \mathfrak{s} \to \mathfrak{f} \stackrel{\pi\circ\rho}{\to} \mathfrak{q}/\mathfrak{a} \to 0$. We know from the proof of Proposition 3.6 that $\mathrm{Im}(C) \cong \frac{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}$, then by Proposition 4.5 we have the following commutative diagram:

$$\begin{tikzcd}
 & & Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}\right) \arrow[r] \arrow[d, hook] & Z^*_{\mathsf{Lie}}(\mathfrak{q}) \arrow[d, hook] & \\
0 \arrow[r] & \frac{\mathfrak{r}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}} \arrow[r] \arrow[ur] & \frac{\mathfrak{f}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}} \arrow[r, "\overline{\rho}"] \arrow[d, two heads, "\epsilon"] & \mathfrak{q} \arrow[r] \arrow[d, two heads, "\gamma"] & 0 \\
 & & \bullet \arrow[r, "\sim"] & \mathfrak{q}/Z^*_{\mathsf{Lie}}(\mathfrak{q}) &
\end{tikzcd}$$

Hence $C = 0 \Leftrightarrow \frac{[\mathfrak{s},\mathfrak{f}]_{\mathsf{Lie}}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}} = 0 \Leftrightarrow \frac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}} \subseteq Z_{\mathsf{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}\right) \Leftrightarrow \gamma \circ \overline{\rho}\left(\frac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}\right) = \epsilon\left(\frac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}\right) = 0 \Leftrightarrow \mathfrak{a} = \rho(\mathfrak{s}) = \overline{\rho}\left(\frac{\mathfrak{s}}{[\mathfrak{r},\mathfrak{f}]_{\mathsf{Lie}}}\right) \subseteq \mathrm{Ker}(\gamma) = Z^*_{\mathsf{Lie}}(\mathfrak{q})$. $\square$

**Corollary 4.10** *For a Leibniz algebra $\mathfrak{q}$ the following statements are equivalent:*

(a) *$\mathfrak{q}$ is* Lie*-unicentral.*

(b) *The map $C : Z_{\mathsf{Lie}}(\mathfrak{q}) \otimes \mathfrak{q}_{\mathsf{Lie}} \to HL_2^{\mathsf{Lie}}(\mathfrak{q})$ in the sequence* (**E**) *associated to the* Lie*-central extension $0 \to Z_{\mathsf{Lie}}(\mathfrak{q}) \to \mathfrak{q} \to \mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q}) \to 0$ is the zero map.*

(c) *The canonical homomorphism $HL_2^{\mathsf{Lie}}(\mathfrak{q}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q}))$ is injective.*

*Proof.* This is a consequence of the exactness of the sequence (**E**) associated to the Lie-central extension $0 \to Z_{\mathsf{Lie}}(\mathfrak{q}) \to \mathfrak{q} \to \mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q}) \to 0$. $\square$

**Remark 4.11** *If $\mathfrak{q}$ is a* Lie*-unicentral Leibniz algebra, by Corollary 4.10 and the sequence* (**E**) *associated to the* Lie*-central extension $0 \to Z_{\mathsf{Lie}}(\mathfrak{q}) \to \mathfrak{q} \to \mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q}) \to 0$, we have*

$$HL_2^{\mathsf{Lie}}(\mathfrak{q}) = \mathrm{Ker}\left(\theta^*(\mathfrak{q}) : HL_2^{\mathsf{Lie}}(\mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q})) \to Z_{\mathsf{Lie}}(\mathfrak{q})\right).$$

# 5 Lie-solvable and Lie-nilpotent Leibniz algebras

In this section, by using the relative commutators, we introduce the notions of Lie-solvability and Lie-nilpotency of Leibniz algebras and investigate their properties.

**Definition 5.1** *Let $\mathfrak{m}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{q}$. A series from $\mathfrak{m}$ to $\mathfrak{q}$ is a finite sequence of two-sided ideals $\mathfrak{m}_i$, $0 \leq i \leq k$, of $\mathfrak{q}$ such that*

$$\mathfrak{m} = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{k-1} \trianglelefteq \mathfrak{m}_k = \mathfrak{q} \ .$$

*$k$ is called the length of this series.*

*A series from $\mathfrak{m}$ to $\mathfrak{q}$ of length $k$ is said to be* Lie*-central (resp.* Lie*-abelian) if $[\mathfrak{m}_i, \mathfrak{q}]_{\mathsf{Lie}} \subseteq \mathfrak{m}_{i-1}$, or equivalently $\mathfrak{m}_i/\mathfrak{m}_{i-1} \subseteq Z_{\mathsf{Lie}}(\mathfrak{q}/\mathfrak{m}_{i-1})$ (resp. if $[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathsf{Lie}} \subseteq \mathfrak{m}_{i-1}$, or equivalently $[\mathfrak{m}_i/\mathfrak{m}_{i-1}, \mathfrak{m}_i/\mathfrak{m}_{i-1}]_{\mathsf{Lie}} = 0$) for $1 \leq i \leq k$ .*

*A series from $0$ to $\mathfrak{q}$ is called a series of the Leibniz algebra $\mathfrak{q}$.*

**Definition 5.2** *A Leibniz algebra $\mathfrak{q}$ is said to be* Lie*-solvable if it has a* Lie*-abelian series. If $k$ is the minimal length of such series, then $k$ is called the class of* Lie*-solvability of $\mathfrak{q}$.*

We show below that among all Lie-abelian series of a Lie-solvable Leibniz algebra there is one which descends most rapidly.

**Definition 5.3** *The* Lie*-derived series of a Leibniz algebra $\mathfrak{q}$ is the sequence*

$$\cdots \trianglelefteq \mathfrak{q}^{(i)} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{(1)} \trianglelefteq \mathfrak{q}^{(0)}$$

*of two-sided ideals of $\mathfrak{q}$ defined inductively by*

$$\mathfrak{q}^{(0)} = \mathfrak{q} \quad \textit{and} \quad \mathfrak{q}^{(i)} = [\mathfrak{q}^{(i-1)}, \mathfrak{q}^{(i-1)}]_{\mathsf{Lie}}, \ i \geq 1.$$

**Theorem 5.4**

*(a) Let $\mathfrak{q}$ be a Leibniz algebra and $\mathfrak{m} = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{j-1} \trianglelefteq \mathfrak{m}_j = \mathfrak{q}$ be a* Lie*-abelian series from $\mathfrak{m}$ to $\mathfrak{q}$, then $\mathfrak{q}^{(i)} \subseteq \mathfrak{m}_{j-i}$, $0 \leq i \leq j$.*

*(b) A Leibniz algebra $\mathfrak{q}$ is* Lie*-solvable with class of* Lie*-solvability $k$ if and only if $\mathfrak{q}^{(k)} = 0$ and $\mathfrak{q}^{(k-1)} \neq 0$.*

*Proof.* (a) This easily follows by induction on $i$.

(b) If $\mathfrak{q}^{(k)} = 0$ and $\mathfrak{q}^{(k-1)} \neq 0$, then $0 = \mathfrak{q}^{(k)} \trianglelefteq \mathfrak{q}^{(k-1)} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{(1)} \trianglelefteq \mathfrak{q}^{(0)} = \mathfrak{q}$ is a Lie-abelian series and by (a) its length is minimal. Therefore $\mathfrak{q}$ is a Lie-solvable with class of solvability $k$. The converse statement directly follows from (a). □

**Example 5.5**

*(a) Abelian Leibniz algebras are* Lie*-solvable Leibniz algebras of class 1.*

*(b) Lie algebras are* Lie*-solvable Leibniz algebras of class 1.*

*(c) The three-dimensional (non-Lie) Leibniz algebra with $\mathbb{K}$-linear basis $\{a_1, a_2, a_3\}$ and Leibniz bracket given by*

$$[a_1, a_3] = [a_2, a_3] = a_2, [a_3, a_3] = a_1$$

*and zero elsewhere (see [5]), is a* Lie*-solvable Leibniz algebra of class 2 and solvable of class 2.*

*(d) The five-dimensional perfect (non-Lie) Leibniz algebra with $\mathbb{K}$-linear basis $\{a_1, a_2, a_3, a_4, a_5\}$ and Leibniz bracket given by*

$$\begin{array}{lll} [a_2, a_1] = -a_3, & [a_1, a_2] = a_3, & [a_1, a_3] = -2a_1, \\ [a_3, a_1] = 2a_1, & [a_3, a_2] = -2a_2, & [a_2, a_3] = 2a_2, \\ [a_5, a_1] = a_4, & [a_4, a_2] = a_5, & [a_4, a_3] = -a_4, \\ & & [a_5, a_3] = a_5, \end{array}$$

*and zero elsewhere (see [22]) is* Lie*-solvable of class 2, but it is not a solvable Leibniz algebra.*

*(e) Subalgebras and images by homomorphisms of* Lie*-solvable Leibniz algebras are* Lie*-solvable as well.*

*(f) Intersection and sum of* Lie*-solvable two-sided ideals of a Leibniz algebra are* Lie*-solvable two-sided ideals as well.*

**Proposition 5.6**

*(a) Let $\mathfrak{n}$ be a* Lie*-solvable ideal of a Leibniz algebra $\mathfrak{q}$ such that $\mathfrak{q}/\mathfrak{n}$ is* Lie*-solvable, then $\mathfrak{q}$ itself is* Lie*-solvable.*

*(b) Let $\mathfrak{m}$ and $\mathfrak{n}$ be* Lie*-solvable two-sided ideals of a Leibniz algebra $\mathfrak{q}$, then $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}}$ is a* Lie*-solvable two-sided ideal of $\mathfrak{q}$.*

*Proof.* (a) Since $\mathfrak{q}/\mathfrak{n}$ is Lie-solvable, then there exists $k \in \mathbb{N}$ such that $0 = (\mathfrak{q}/\mathfrak{n})^{(k)} = \mathfrak{q}^{(k)}/\mathfrak{n}$, that is $\mathfrak{q}^{(k)} = \mathfrak{n}$. On the other hand, since $\mathfrak{n}$ is Lie-solvable, then there exists $j \in \mathbb{N}$ such that $\mathfrak{n}^{(j)} = 0$. Hence $\mathfrak{q}^{(k+j)} = \left(\mathfrak{q}^{(k)}\right)^{(j)} = \mathfrak{n}^{(j)} = 0$.

(b) By Lemma 3.3 $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}}$ is a two-sided ideal of $\mathfrak{q}$. Moreover, $[\mathfrak{m}, \mathfrak{n}]_{\mathsf{Lie}} \subseteq \mathfrak{m} \cap \mathfrak{n}$. Then the statements (e) and (f) in Example 5.5 complete the proof. □

**Definition 5.7** *A Leibniz algebra $\mathfrak{q}$ is said to be* Lie*-nilpotent if it has a* Lie*-central series. If $k$ is the minimal length of such series, then $k$ is called the class of* Lie*-nilpotency of $\mathfrak{q}$.*

We show below that among all $\mathsf{Lie}$-central series of a $\mathsf{Lie}$-nilpotent Leibniz algebra there is one which descends most rapidly.

**Definition 5.8** *The lower* $\mathsf{Lie}$*-central series of a Leibniz algebra* $\mathfrak{q}$ *is the sequence*

$$\cdots \trianglelefteq \mathfrak{q}^{[i]} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{[2]} \trianglelefteq \mathfrak{q}^{[1]}$$

*of two-sided ideals of* $\mathfrak{q}$ *defined inductively by*

$$\mathfrak{q}^{[1]} = \mathfrak{q} \quad \textit{and} \quad \mathfrak{q}^{[i]} = [\mathfrak{q}^{[i-1]}, \mathfrak{q}]_{\mathsf{Lie}}, \ i \geq 2.$$

**Theorem 5.9**

*(a) Let* $\mathfrak{q}$ *be a Leibniz algebra and* $0 = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{j-1} \trianglelefteq \mathfrak{m}_j = \mathfrak{q}$ *be a* $\mathsf{Lie}$*-central series of* $\mathfrak{q}$*, then* $\mathfrak{q}^{[i]} \subseteq \mathfrak{m}_{j-i+1}$*,* $1 \leq i \leq j+1$*.*

*(b) A Leibniz algebra* $\mathfrak{q}$ *is* $\mathsf{Lie}$*-nilpotent with class of nilpotency* $k$ *if and only if* $\mathfrak{q}^{[k+1]} = 0$ *and* $\mathfrak{q}^{[k]} \neq 0$*.*

*Proof.* (a) This follows by induction on $i$.

(b) If $\mathfrak{q}^{[k+1]} = 0$ and $\mathfrak{q}^{[k]} \neq 0$, then $0 = \mathfrak{q}^{[k+1]} \trianglelefteq \mathfrak{q}^{[k]} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{[2]} \trianglelefteq \mathfrak{q}^{[1]} = \mathfrak{q}$ is a $\mathsf{Lie}$-central series, which has minimal length by (a). Therefore $\mathfrak{q}$ is $\mathsf{Lie}$-nilpotent of class $k$. The inverse statement directly follows from (a). □

**Definition 5.10** *The upper* $\mathsf{Lie}$*-central series of a Leibniz algebra* $\mathfrak{q}$ *is the sequence of two-sided ideals*

$$\mathcal{Z}_0^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_1^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \cdots$$

*defined inductively by*

$$\mathcal{Z}_0^{\mathsf{Lie}}(\mathfrak{q}) = 0 \quad \textit{and} \quad \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q}) = C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{q}, \mathcal{Z}_{i-1}^{\mathsf{Lie}}(\mathfrak{q})), \ i \geq 1.$$

Let us observe that $\mathcal{Z}_1^{\mathsf{Lie}}(\mathfrak{q}) = Z_{\mathsf{Lie}}(\mathfrak{q})$ and, by Lemma 3.3, $\mathcal{Z}_i^{\mathrm{Lie}}(\mathfrak{q})$ indeed is a two-sided ideal of $\mathfrak{q}$.

**Lemma 5.11** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be two-sided ideals of a Leibniz algebra* $\mathfrak{q}$*. If* $[\mathfrak{m}, \mathfrak{q}]_{\mathsf{Lie}} \subseteq \mathfrak{n}$*, then* $\mathfrak{m} \subseteq C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{q}, \mathfrak{n})$*.*

*Proof.* This is straightforward. □

Now we show that among all $\mathsf{Lie}$-central series of a nilpotent Leibniz algebra there is one which ascends most rapidly.

**Proposition 5.12** *Let* $0 = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{k-1} \trianglelefteq \mathfrak{m}_k = \mathfrak{q}$ *be a* $\mathsf{Lie}$*-central series of a* $\mathsf{Lie}$*-nilpotent Leibniz algebra* $\mathfrak{q}$*, then* $\mathfrak{m}_i \subseteq \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q})$*,* $0 \leq i \leq k$*.*

*Proof.* Obviously the assertion holds for $i = 0$. Proceeding by induction on $i$, we assume that the assertion is true for $i-1$, then using the $\mathsf{Lie}$-centrality and Lemma 5.11, we have $\mathfrak{m}_i \subseteq C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{q}, \mathfrak{m}_{i-1}) \subseteq C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{q}, \mathcal{Z}_{i-1}^{\mathsf{Lie}}(\mathfrak{q})) = \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q})$. □

**Theorem 5.13** *A Leibniz algebra* $\mathfrak{q}$ *is* $\mathsf{Lie}$*-nilpotent with class of* $\mathsf{Lie}$*-nilpotency* $k$ *if and only if* $\mathcal{Z}_k^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{q}$ *and* $\mathcal{Z}_{k-1}^{\mathsf{Lie}}(\mathfrak{q}) \neq \mathfrak{q}$.

*Proof.* If $\mathfrak{q}$ is a $\mathsf{Lie}$-nilpotent Leibniz algebra with class of $\mathsf{Lie}$-nilpotency $k$, then Proposition 5.12 implies that $\mathfrak{q} = \mathfrak{m}_k \subseteq \mathcal{Z}_k^{\mathsf{Lie}}(\mathfrak{q}) \subseteq \mathfrak{q}$. Moreover, in this case $0 = \mathcal{Z}_0^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_1^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_{k-1}^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_k^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{q}$ is a $\mathsf{Lie}$-central series of length $k$ of $\mathfrak{q}$. Hence $\mathcal{Z}_{k-1}^{\mathsf{Lie}}(\mathfrak{q}) \neq \mathfrak{q}$.
Conversely, if $\mathcal{Z}_k^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{q}$ and $\mathcal{Z}_{k-1}^{\mathsf{Lie}}(\mathfrak{q}) \neq \mathfrak{q}$ then $0 = \mathcal{Z}_0^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_1^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_{k-1}^{\mathsf{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_k^{\mathsf{Lie}}(\mathfrak{q}) = \mathfrak{q}$ is a $\mathsf{Lie}$-central series of $\mathfrak{q}$ and by Proposition 5.12 its length is minimal. □

**Example 5.14**

(a) *Abelian Leibniz algebras are* $\mathsf{Lie}$*-nilpotent Leibniz algebras of class 1.*

(b) *Lie algebras are* $\mathsf{Lie}$*-nilpotent Leibniz algebras of class 1.*

(c) *The three-dimensional non-Lie Leibniz algebra with* $\mathbb{K}$*-linear basis* $\{a_1, a_2, a_3\}$ *and Leibniz bracket given by*

$$[a_3, a_3] = a_1$$

*and zero elsewhere (see [5]), is a* $\mathsf{Lie}$*-nilpotent Leibniz algebra of class 2.*

(d) *The non-Lie Leibniz algebra given in Example 5.5 (c) is non* $\mathsf{Lie}$*-nilpotent.*

(e) *The four-dimensional (non-Lie) Leibniz algebra with* $\mathbb{K}$*-linear basis* $\{a_1, a_2, a_3, a_4\}$ *and Leibniz bracket given by*

$$[e_1, e_1] = e_3, [e_2, e_4] = e_2, [e_4, e_2] = -e_2$$

*and zero elsewhere (see [4]) is* $\mathsf{Lie}$*-nilpotent of class 2, but it is not a nilpotent Leibniz algebra.*

(f) *Subalgebras and images by homomorphisms of* $\mathsf{Lie}$*-nilpotent Leibniz algebras are* $\mathsf{Lie}$*-nilpotent Leibniz algebras.*

(g) *Intersection and sum of* $\mathsf{Lie}$*-nilpotent two-sided ideals of a Leibniz algebra are* $\mathsf{Lie}$*-nilpotent two-sided ideals as well.*

**Proposition 5.15**

(a) *If* $\mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q})$ *is a* $\mathsf{Lie}$*-nilpotent Leibniz algebra, then* $\mathfrak{q}$ *is a* $\mathsf{Lie}$*-nilpotent Leibniz algebra.*

*(b) If $\mathfrak{q}$ is a $\mathsf{Lie}$-nilpotent and non trivial Leibniz algebra, then $Z_{\mathsf{Lie}}(\mathfrak{q}) \neq 0$.*

*(c) If $\mathfrak{g} \twoheadrightarrow \mathfrak{q}$ is a $\mathsf{Lie}$-central extension of a $\mathsf{Lie}$-nilpotent Leibniz algebra $\mathfrak{q}$, then $\mathfrak{g}$ is $\mathsf{Lie}$-nilpotent as well.*

*(d) A $\mathsf{Lie}$-nilpotent Leibniz algebra is $\mathsf{Lie}$-solvable as well.*

*Proof.* (a) There exists $k \in \mathbb{N}$ such that $(\mathfrak{q}/Z_{\mathsf{Lie}}(\mathfrak{q}))^{[k]} = 0$, then $\mathfrak{q}^{[k]} \subseteq Z_{\mathsf{Lie}}(\mathfrak{q})$, hence $\mathfrak{q}^{[k+1]} \subseteq [Z_{\mathsf{Lie}}(\mathfrak{q}), \mathfrak{q}]_{\mathsf{Lie}} = 0$.

(b) Assume that $\mathfrak{q}$ has $\mathsf{Lie}$-nilpotency class equal to $k$, that is $[\mathfrak{q}^{[k]}, \mathfrak{q}]_{\mathsf{Lie}} = \mathfrak{q}^{[k+1]} = 0$, then Lemma 5.11 implies that $0 \neq \mathfrak{q}^{[k]} \subseteq C_{\mathfrak{q}}^{\mathsf{Lie}}(\mathfrak{q}, 0) = Z_{\mathsf{Lie}}(\mathfrak{q})$.

(c) There exists $k \in \mathbb{N}$ such that $\mathfrak{q}^{[k]} = 0$. Then $(\mathfrak{g}/\mathfrak{n})^{[k]} = \mathfrak{g}^{[k]}/\mathfrak{n} = 0$, where $\mathfrak{n} = \mathrm{Ker}(\mathfrak{g} \twoheadrightarrow \mathfrak{q})$. Hence $\mathfrak{g}^{[k]} \subseteq \mathfrak{n} \subseteq Z_{\mathsf{Lie}}(\mathfrak{g})$ and $\mathfrak{g}^{[k+1]} = [\mathfrak{g}^{[k]}, \mathfrak{g}]_{\mathsf{Lie}} = 0$.

(d) By induction on $i$, it is easy to see that $\mathfrak{q}^{(i)} \subseteq \mathfrak{q}^{[i+1]}$, $i \geq 0$. $\square$

# 6 Homological criterion for $\mathsf{Lie}$-nilpotency

**Definition 6.1** *Let $\mathfrak{n}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{q}$. The lower $\mathsf{Lie}$-central series of $\mathfrak{q}$ relative to $\mathfrak{n}$ is the sequence*

$$\cdots \trianglelefteq \mathfrak{n}^{[i]} \trianglelefteq \cdots \trianglelefteq \mathfrak{n}^{[2]} \trianglelefteq \mathfrak{n}^{[1]}$$

*of two-sided ideals of $\mathfrak{q}$ defined inductively by*

$$\mathfrak{n}^{[1]} = \mathfrak{n} \quad \textit{and} \quad \mathfrak{n}^{[i]} = [\mathfrak{n}^{[i-1]}, \mathfrak{q}]_{\mathsf{Lie}}, \quad i \geq 2.$$

Note that $[\mathfrak{n}^{[i]}/\mathfrak{n}^{[i+1]}, \mathfrak{n}^{[i]}/\mathfrak{n}^{[i+1]}]_{\mathsf{Lie}} = 0$. When $\mathfrak{n} = \mathfrak{q}$ we obtain Definition 5.8. If $\varphi : \mathfrak{g} \to \mathfrak{q}$ is a homomorphism of Leibniz algebras such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{m}$ is a two-sided ideal of $\mathfrak{g}$ and $\mathfrak{n}$ is a two-sided ideal of $\mathfrak{q}$, then $\varphi(\mathfrak{m}^{[i]}) \subseteq \mathfrak{n}^{[i]}$, $i \geq 1$.

**Theorem 6.2** *Let $\varphi : \mathfrak{g} \to \mathfrak{q}$ be a homomorphism of Leibniz algebras such that $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{m}$ is a two-sided ideal of $\mathfrak{g}$ and $\mathfrak{n}$ is a two-sided ideal of $\mathfrak{q}$, and the following properties hold:*

*(a) the induced homomorphism $HL_1^{\mathsf{Lie}}(\mathfrak{g}) \to HL_1^{\mathsf{Lie}}(\mathfrak{q})$ is an isomorphism;*

*(b) the induced homomorphism $HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q})$ is an epimorphism;*

*(c) the induced homomorphism $\varphi_1 : \mathfrak{g}/\mathfrak{m} \to \mathfrak{q}/\mathfrak{n}$ is an isomorphism.*

*Then $\varphi$ induces a natural isomorphism $\varphi_k : \mathfrak{g}/\mathfrak{m}^{[k]} \to \mathfrak{q}/\mathfrak{n}^{[k]}$, $k \geq 1$.*

*Proof.* We prove by induction on $k$. For $k=1$ we have the statement (c). Suppose the theorem is true for $k-1$. Applying sequence (**D**), which is natural [12], to the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^{[k-1]} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, "\varphi"] & \mathfrak{g}/\mathfrak{m}^{[k-1]} \arrow[r] \arrow[d, "\wr" left, "\varphi_{k-1}"] & 0 \\
0 \arrow[r] & \mathfrak{n}^{[k-1]} \arrow[r] & \mathfrak{q} \arrow[r] & \mathfrak{q}/\mathfrak{n}^{[k-1]} \arrow[r] & 0
\end{tikzcd}$$

we get the following commutative diagram

$$\begin{tikzcd}
HL_2^{\mathsf{Lie}}(\mathfrak{g}) \arrow[r] \arrow[d] & HL_2^{\mathsf{Lie}}(\frac{\mathfrak{g}}{\mathfrak{m}^{[k-1]}}) \arrow[r] \arrow[d] & \frac{\mathfrak{m}^{[k-1]}}{[\mathfrak{m}^{[k-1]},\mathfrak{g}]_{\mathsf{Lie}}} \arrow[r] \arrow[d] & HL_1^{\mathsf{Lie}}(\mathfrak{g}) \arrow[r] \arrow[d] & HL_1^{\mathsf{Lie}}(\frac{\mathfrak{g}}{\mathfrak{m}^{[k-1]}}) \arrow[r] \arrow[d] & 0 \\
HL_2^{\mathsf{Lie}}(\mathfrak{q}) \arrow[r] & HL_2^{\mathsf{Lie}}(\frac{\mathfrak{q}}{\mathfrak{n}^{[k-1]}}) \arrow[r] & \frac{\mathfrak{n}^{[k-1]}}{[\mathfrak{n}^{[k-1]},\mathfrak{q}]_{\mathsf{Lie}}} \arrow[r] & HL_1^{\mathsf{Lie}}(\mathfrak{q}) \arrow[r] & HL_1^{\mathsf{Lie}}(\frac{\mathfrak{q}}{\mathfrak{n}^{[k-1]}}) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, which holds in a semi-abelian category [1, 21], we get $\frac{\mathfrak{m}^{[k-1]}}{[\mathfrak{m}^{[k-1]},\mathfrak{g}]_{\mathsf{Lie}}} \cong \frac{\mathfrak{n}^{[k-1]}}{[\mathfrak{n}^{[k-1]},\mathfrak{q}]_{\mathsf{Lie}}}$, i. e. $\frac{\mathfrak{m}^{[k-1]}}{\mathfrak{m}^{[k]}} \cong \frac{\mathfrak{n}^{[k-1]}}{\mathfrak{n}^{[k]}}$. Then the short Five Lemma applied to the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{\mathfrak{m}^{[k-1]}}{\mathfrak{m}^{[k]}} \arrow[r] \arrow[d] & \frac{\mathfrak{g}}{\mathfrak{m}^{[k]}} \arrow[r] \arrow[d] & \frac{\mathfrak{g}}{\mathfrak{m}^{[k-1]}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\mathfrak{n}^{[k-1]}}{\mathfrak{n}^{[k]}} \arrow[r] & \frac{\mathfrak{q}}{\mathfrak{n}^{[k]}} \arrow[r] & \frac{\mathfrak{q}}{\mathfrak{n}^{[k-1]}} \arrow[r] & 0
\end{tikzcd}$$

and the induction complete the proof. □

**Corollary 6.3** *Let $\varphi : \mathfrak{g} \to \mathfrak{q}$ be a homomorphism of Leibniz algebras such that $\varphi_{\mathrm{Lie}} : \mathfrak{g}_{\mathrm{Lie}} \to \mathfrak{q}_{\mathrm{Lie}}$ is an isomorphism and the induced homomorphism $HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q})$ is an epimorphism. If $\mathfrak{g}$ and $\mathfrak{q}$ are* Lie*-nilpotent Leibniz algebras, then $\varphi$ is an isomorphism.*

*Proof.* Take $\mathfrak{m} = \mathfrak{g}$ and $\mathfrak{n} = \mathfrak{q}$ in Theorem 6.2. Then the assertion follows by keeping in mind that $HL_1^{\mathsf{Lie}}(\mathfrak{g}) \cong \mathfrak{g}_{\mathrm{Lie}}$, $HL_1^{\mathsf{Lie}}(\mathfrak{q}) \cong \mathfrak{q}_{\mathrm{Lie}}$ and there exists $k \geq 1$ such that $\mathfrak{g}^{[k]} = \mathfrak{q}^{[k]} = 0$. □

**Corollary 6.4** *Suppose $\mathfrak{q}$ and $\mathfrak{q}_{\mathrm{Lie}}$ both are* Lie*-nilpotent Leibniz algebras. Then necessarily $\mathfrak{q} \cong \mathfrak{q}_{\mathrm{Lie}}$, that is, $\mathfrak{q}$ is a Lie algebra.*

*Proof.* This follows by applying Corollary 6.3 to the canonical epimorphism $\mathfrak{q} \twoheadrightarrow \mathfrak{q}_{\mathrm{Lie}}$. □

**Lemma 6.5** *Let $\mathfrak{n}$ be a two-sided ideal of a Leibniz algebra $\mathfrak{q}$. The lower* Lie*-central series determined by $\mathfrak{n}$ vanishes if and only if there exists $i \geq 0$ such that $\mathfrak{n} \subseteq \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q})$.*

*Proof.* It is enough to use the following obvious equivalence:

$$\mathfrak{n}^{[i]} \subseteq \mathcal{Z}_k^{\mathsf{Lie}}(\mathfrak{q}) \Leftrightarrow \mathfrak{n}^{[i+1]} \subseteq \mathcal{Z}_{k-1}^{\mathsf{Lie}}(\mathfrak{q}).$$

$\square$

**Theorem 6.6** *Let* $\mathfrak{m}$ *be a two-sided ideal of* $\mathfrak{g}$ *and* $\mathfrak{n}$ *be a two-sided ideal of* $\mathfrak{q}$ *such that* $\mathfrak{m} \subseteq \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{g})$ *and* $\mathfrak{n} \subseteq \mathcal{Z}_j^{\mathsf{Lie}}(\mathfrak{q})$, $i, j \geq 0$. *Let* $\varphi : \mathfrak{g} \to \mathfrak{q}$ *be a homomorphism of Leibniz algebras such that* $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ *and satisfying the following conditions:*

*(a) the induced homomorphism* $HL_1^{\mathsf{Lie}}(\mathfrak{g}) \to HL_1^{\mathsf{Lie}}(\mathfrak{q})$ *is an isomorphism;*

*(b) the induced homomorphism* $HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q})$ *is an epimorphism;*

*(c) the induced homomorphism* $\mathfrak{g}/\mathfrak{m} \to \mathfrak{q}/\mathfrak{n}$ *is an isomorphism.*

*Then* $\varphi : \mathfrak{g} \to \mathfrak{q}$ *is an isomorphism.*

*Proof.* This is a consequence of Theorem 6.2 and Lemma 6.5. $\square$

**Corollary 6.7** *Let* $\varphi : \mathfrak{g} \to \mathfrak{q}$ *be a homomorphism of Leibniz algebras such that* $\varphi(\mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{g})) \subseteq \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q})$ *for any* $i \geq 0$ *and satisfying the following conditions:*

*(a) the induced homomorphism* $HL_1^{\mathsf{Lie}}(\mathfrak{g}) \to HL_1^{\mathsf{Lie}}(\mathfrak{q})$ *is an isomorphism;*

*(b) the induced homomorphism* $HL_2^{\mathsf{Lie}}(\mathfrak{g}) \to HL_2^{\mathsf{Lie}}(\mathfrak{q})$ *is an epimorphism;*

*(c) the induced homomorphism* $\mathfrak{g}/\mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{g}) \to \mathfrak{q}/\mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q})$ *is an isomorphism.*

*Then* $\varphi : \mathfrak{g} \to \mathfrak{q}$ *is an isomorphism.*

*Proof.* This follows by applying Theorem 6.6 to the case $\mathfrak{m} = \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{g})$ and $\mathfrak{n} = \mathcal{Z}_i^{\mathsf{Lie}}(\mathfrak{q})$. $\square$


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